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A block completion problem for matrix-valued inner functions[1]

B. Fritzsche[a], B. Kirstein[a], K. Müller[b,*]

[a] *Mathematisches Institut, Universität Leipzig, Augustusplatz 10, D-04109 Leipzig, Germany*
[b] *Department Computerwetenschappen, Katholieke Universiteit Leuven, Celestijnenlaan 200 A, B-3001 Leuven, Heverlee, Belgium*

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Abstract

In this paper, several block completion problems for matrix-valued inner functions are studied.

0. Introduction

In this paper, we will study particular block completion problems for matrix-valued inner functions. Given a $p \times q$ Schur function f defined on $\mathbb{D}$ we will describe the set of all $(p+q) \times (p+q)$ inner functions the right upper $p \times q$ block of which coincides with f. This problem arose from Darlington synthesis. Namely, using the Potapov–Ginzburg transform it is easily checked that f admits a canonical Darlington realization with the aid of some j_{pq}-inner function if and only if f can be embedded into the right upper $p \times q$ corner of some $(p+q) \times (p+q)$ inner function. The investigation of such completion problems for matrix-valued inner functions originates in [12], where the solvability was characterized. A detailed analysis of the structure of the set of all solutions started with Arov's paper [4] which can be considered as a continuation of his previous work on Darlington synthesis (see [1–3]). In the context of generalized bitangential Schur–Nevanlinna interpolation (see [6, 7]), new subclasses of inner matrix-valued functions became interesting from several point of view. In Section 5, we will discuss the above-mentioned block completion problem for some of these subclasses. Observe that our methods are mainly based on pseudocontinuation and spectral factorization techniques as developed in [5, 8–10].

* Corresponding author. E-mail: karsten.muller@cs.kuleuven.ac.be.

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1. Some basic facts on various classes of meromorphic matrix-valued functions

In this section, we will summarize some facts on several classes of meromorphic functions. For a detailed treatment, we refer the reader to the monographs in [22, 16]. We will start with some notation. Throughout this paper, let p and q be positive integers. We will use $\mathbb{C}$, $\mathbb{D}$, $\mathbb{T}$, $\mathbb{C}_0$ and $\mathbb{E}$ to denote the set of complex numbers, the open unit disc, the unit circle, the extended complex plane and the exterior of the closed unit disc, respectively,

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{C}_0 := \mathbb{C} \cup \{\infty\}, \quad \mathbb{E} := \mathbb{C}_0 \backslash (\mathbb{D} \cup \mathbb{T}).$$

The linear Lebesgue–Borel measure on $\mathbb{T}$ will be designated by $\underline{\lambda}$. Let $\mathfrak{B}$ be the complex linear space of all Borel measurable mappings $\Phi : \mathbb{T} \to \mathbb{C}^{p\times q}$. Then $\mathfrak{Z} := \{\Phi \in \mathfrak{B} : \underline{\lambda}(\{w \in \mathbb{T} : \Phi(w) \neq 0_{p\times q}\}) = 0\}$ is a linear subspace of $\mathfrak{B}$. In the following we will deal with the quotient space $\mathfrak{Q} := \mathfrak{B}/\mathfrak{Z}$. If $\Phi \in \mathfrak{B}$, then $\langle \Phi \rangle$ denotes that element of $\mathfrak{Q}$ which is generated by Φ. Obviously, the relation $\langle \Phi \rangle = \langle \Psi \rangle$ is satisfied if and only if $\Phi(w) = \Psi(w)$ for $\underline{\lambda}$-almost all $w \in \mathbb{T}$.

Assume that G is a simply connected domain of $\mathbb{C}_0$. Then let $\mathcal{NM}(G)$ be the *Nevanlinna class* of all functions which are meromorphic in G and which can be represented as a quotient of two bounded holomorphic functions in G. If $g \in \mathcal{NM}(\mathbb{D})$ (respectively, $g \in \mathcal{NM}(\mathbb{E})$), then a well-known theorem due to Fatou implies that there exist a Borelian subset B_0 of the unit circle $\mathbb{T}$ with $\underline{\lambda}(B_0) = 0$ and a Borel measurable function $\underline{g} : \mathbb{T} \to \mathbb{C}$ such that

$$\lim_{r\to 1-0} g(rz) = \underline{g}(z) \quad \text{(respectively, } \lim_{r\to 1+0} g(rz) = \underline{g}(z))$$

for all $z \in \mathbb{T} \backslash B_0$. In the sequel, we will continue to use the symbol $\underline{g}$ to denote such a radial boundary function of a function g which belongs to $\mathcal{NM}(\mathbb{D})$ or $\mathcal{NM}(\mathbb{E})$.

Let $g \in \mathcal{NM}(\mathbb{D})$. Then one says that g *admits a pseudocontinuation* (*into* $\mathbb{E}$) if there exists a function $g^\# \in \mathcal{NM}(\mathbb{E})$ such that the radial boundary values $\underline{g}$ and $\underline{g^\#}$ of g and $g^\#$, respectively, coincide $\underline{\lambda}$-almost everywhere on $\mathbb{T}$. It is obvious that a function $g \in \mathcal{NM}(\mathbb{D})$ admits at most one pseudocontinuation. Note that if $g \in \mathcal{NM}(\mathbb{D})$ admits a pseudocontinuation $g^\#$ and if, additionally, g is analytically continuable through some open arc of $\mathbb{T}$, then the analytic continuation coincides with the pseudocontinuation. Later, we will use some properties of pseudocontinuation which can be found, e.g., in [23, Lecture 2; 13] et al. In the following, the notation $\Pi(\mathbb{D})$ stands for the set of all functions $g \in \mathcal{NM}(\mathbb{D})$ which admit a pseudocontinuation. If $g \in \Pi(\mathbb{D})$, then the symbol $g^\#$ will be used to denote the pseudocontinuation of g.

The subalgebra of all $g \in \mathcal{NM}(G)$ which are holomorphic in G will be denoted by $\mathcal{N}(G)$. The class $\mathcal{N}(G)$ can be described as the set of all functions g which are holomorphic in $\mathbb{D}$ and which fulfill

$$\sup_{r\in[0,1)} \frac{1}{2\pi} \int_{\mathbb{T}} \log^+ |g(rz)| \underline{\lambda}(\mathrm{d}z) < \infty,$$

where $\log^+ x := \max(\log x, 0)$ for each $x \in [0, \infty)$. If $g : \mathbb{D} \to \mathbb{C}$ admits a representation

$$g(w) = \alpha \exp\left\{\frac{1}{2\pi} \int_{\mathbb{T}} \frac{z+w}{z-w} \log k(z) \underline{\lambda}(\mathrm{d}z)\right\}, \quad w \in \mathbb{D},$$

with some $\alpha \in \mathbb{T}$ and some Borel measurable function $k : \mathbb{T} \to [0,\infty)$ which satisfies $(1/2\pi)\int_{\mathbb{T}} |\log k|\, \mathrm{d}\underline{\lambda} < \infty$, then g belongs to $\mathcal{N}(\mathbb{D})$. Such functions g are called *outer*. For all $g \in \mathcal{N}(\mathbb{D})$, the inequality

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log^+ |\underline{g}(z)|\underline{\lambda}(\mathrm{d}z) \leqslant \lim_{r\to 1-0} \frac{1}{2\pi}\int_{\mathbb{T}} \log^+ |g(rz)|\underline{\lambda}(\mathrm{d}z) \tag{1}$$

holds true. By the *Smirnov class* $\mathcal{N}_+(\mathbb{D})$ we will mean the set of all $g \in \mathcal{N}(\mathbb{D})$ for which equality holds true in (1). The class $\mathcal{N}_+(\mathbb{D})$ proves to be a subalgebra of $\mathcal{N}(\mathbb{D})$. If g is outer in $\mathcal{N}(\mathbb{D})$, then g necessarily belongs to $\mathcal{N}_+(\mathbb{D})$. Let $t \in (0,\infty)$. We will use $H^t(\mathbb{D})$ to denote the *Hardy class* of all holomorphic functions $g : \mathbb{D} \to \mathbb{C}$ which fulfill

$$\sup_{r\in[0,1)} \frac{1}{2\pi}\int_{\mathbb{T}} |g(rz)|^t \underline{\lambda}(\mathrm{d}z) < \infty.$$

The notation $H^\infty(\mathbb{D})$ stands for the set of all bounded holomorphic functions. If $0 < t < s \leqslant \infty$ then the relation $H^s(\mathbb{D}) \subset H^t(\mathbb{D}) \subset \mathcal{N}_+(\mathbb{D}) \subset \mathcal{N}(\mathbb{D}) \subset \mathcal{NM}(\mathbb{D})$ holds true.

If $\mathfrak{X}$ is one of the classes $\mathcal{NM}(G)$, $\mathcal{N}(G)$, $\mathcal{N}_+(\mathbb{D})$, $\Pi(\mathbb{D})$ or $H^t(\mathbb{D})$, where $t \in (0,\infty]$, then $\mathfrak{X}^{p\times q}$ designates the class of all $p \times q$ matrix-valued functions each entry of which belongs to $\mathfrak{X}$. If $g = (g_{jk})_{j=1,\dots,p,k=1,\dots,q}$ belongs to $[\Pi(\mathbb{D})]^{p\times q}$, then we will also say that *g admits a pseudocontinuation*. In this case, we will write $g^\#$ for $(g^\#_{jk})_{j=1,\dots,p,k=1,\dots,q}$ and call $g^\#$ *the pseudocontinuation of g*. Let $\mathfrak{X}$ be a nonempty subset of the extended complex plane $\mathbb{C}_0$ and let $f : \mathfrak{X} \to \mathbb{C}^{p\times q}$. Then we will use the symbol $\widehat{f}$ for the function $\widehat{f} : \widehat{\mathfrak{X}} \to \mathbb{C}^{q\times p}$ which is given by $\widehat{\mathfrak{X}} := \{z \in \mathbb{C}_0 : 1/\bar{z} \in \mathfrak{X}\}$ and $\widehat{f}(z) := [f(1/\bar{z})]^*$.

Remark 1.1. If f belongs to $[\mathcal{NM}(\mathbb{D})]^{p\times q}$ (respectively, $[\mathcal{NM}(\mathbb{E})]^{p\times q}$), then $\widehat{f} \in [\mathcal{NM}(\mathbb{E})]^{q\times p}$ (respectively, $\widehat{f} \in [\mathcal{NM}(\mathbb{D})]^{q\times p}$), and $\underline{f}^*$ is a radial boundary function of $\widehat{f}$.

For convience of the reader, we will recall some facts on outer functions which belong to the matricial Smirnov class. A function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is called *outer* (in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$) if $\det \Phi$ is outer in $\mathcal{N}(\mathbb{D})$. An outer function $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ is called *normalized* if $\Phi(0)$ is nonnegative Hermitian. The next result on matrix-valued outer functions which is due to Arov [3] is essential for the further considerations.

Remark 1.2. If $\Phi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ and $\Psi \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$ are outer functions, then the product $\Phi\Psi$ is also an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. If Φ is an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$, then $\det \Phi(w) \neq 0$ for all $w \in \mathbb{D}$ and Φ^{-1} is also an outer function in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$. Conversely, if $\Phi \in [\mathcal{N}(\mathbb{D})]^{q\times q}$ satisfies $\det \Phi(w) \neq 0$ for all $w \in \mathbb{D}$ and if $\Phi^{-1} \in [\mathcal{N}(\mathbb{D})]^{q\times q}$, then Φ and Φ^{-1} are necessarily outer functions in $[\mathcal{N}_+(\mathbb{D})]^{q\times q}$.

2. On matrix-valued Schur functions and inner functions

In this section, we will deal with a class of matrix-valued functions which are contractive and holomorphic in the unit disc. Assume that G is a simply connected domain of $\mathbb{C}_0$. A function $f : G \to \mathbb{C}^{p\times q}$ is said to be *contractive* if $I_p - f(w)f^*(w)$ is nonnegative Hermitian for all $w \in G$. If f is both contractive and holomorphic in G, then f is called a *$p \times q$ Schur function* in G. The set of all $p \times q$ Schur functions will be denoted by $\mathcal{S}_{p\times q}(G)$. If f belongs to $\mathcal{S}_{p\times q}(G)$, then the rank of the matrix $I_p - f(w)f^*(w)$ is constant for all $w \in G$ (see, e.g., [14, Lemma 2.1.5]). This rank is also called the *Schur rank* of f. The following result which was proved in [2] shows an interesting connection between Schur functions and outer functions.

Proposition 2.1. *Let $f \in \mathcal{S}_{q\times q}(\mathbb{D})$ such that $\det(I_q + f)$ does not identically vanish in $\mathbb{D}$. Then $\det(I_q + f)$ is an outer function in $H^\infty(\mathbb{D})$. In particular, $\det(I_q + f(w)) \neq 0$ for all $w \in \mathbb{D}$, and $\frac{1}{2}(I_q + f)$ is an outer function in $\mathcal{S}_{q\times q}(\mathbb{D})$.*

A Schur function $f \in \mathcal{S}_{q\times q}(\mathbb{D})$ is called an *inner function* if f has unitary boundary values $\underline{\lambda}$-almost everywhere on $\mathbb{T}$.

Remark 2.2. Every inner function $f \in \mathcal{S}_{q\times q}(\mathbb{D})$ admits a pseudocontinuation $f^\#$, namely $f^\# = \widehat{f^{-1}}$.

The concept of inner and outer functions was introduced in [11] in the scalar case. A matricial generalization of this concept was created in the context of predicion theory for multivariate stationary sequences ([18–21, 24–26, 28]). These authors obtained the following generalization of the *inner–outer factorization* for the Hardy space $[H^2(\mathbb{D})]^{q\times q}$. Note that a scalar version of this theorem goes back to Smirnov [27].

Theorem 2.3. *Let $\Theta \in [H^2(\mathbb{D})]^{q\times q}$ be such that $\det \Theta$ does not identically vanish in $\mathbb{D}$. Then there exist unique inner functions $V, U \in \mathcal{S}_{q\times q}(\mathbb{D})$ and unique normalized outer functions $\Phi, \Psi \in [H^2(\mathbb{D})]^{q\times q}$ such that $\Theta = V\Phi$ and $\Theta = \Psi U$.*

3. About spectral factorization of pseudocontinuable functions

Let $\Lambda : \mathbb{T} \to \mathbb{C}_{\geqslant}^{q\times q}$ be a Lebesgue integrable function on $\mathbb{T}$. Then a function $\Phi \in [H^2(\mathbb{D})]^{q\times q}$ is called *a left* (respectively, *right*) *spectral factor of* $\langle \Lambda \rangle$ if $\langle \underline{\Phi}\underline{\Phi}^* \rangle = \langle \Lambda \rangle$ (respectively, $\langle \underline{\Phi}^* \underline{\Phi} \rangle = \langle \Lambda \rangle$) is fulfilled.

The next theorem we will use in the following was proved in [28, Theorem 7.7,19] in the context of prediction theory of discrete stationary stochastic processes.

Theorem 3.1. *Let $\Lambda : \mathbb{T} \to \mathbb{C}_{\geqslant}^{q\times q}$ be a Lebesgue integrable function on $\mathbb{T}$ which satisfies*

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log \det \Lambda \, \mathrm{d}\underline{\lambda} > -\infty.$$

Then the following statements hold true:

(a) *There exist a unique normalized outer left spectral factor Φ_0 of $\langle \Lambda \rangle$ and a unique normalized outer right spectral factor Ψ_0 of $\langle \Lambda \rangle$.*

(b) *Let $V \in \mathbb{C}^{q\times q}$ be an arbitrary unitary matrix. Then $\Phi := \Phi_0 V$ (respectively, $\Psi := V\Psi_0$) is an outer left* (*respectively, right*) *spectral factor of $\langle \Lambda \rangle$.*

(c) *Let Φ* (*respectively, Ψ*) *be an arbitrary outer left* (*respectively, right*) *spectral factor of $\langle \Lambda \rangle$. Then there is a unique unitary matrix $V \in \mathbb{C}^{q\times q}$ such that $\Phi = \Phi_0 V$* (*respectively, $\Psi = V\Psi_0$*).

Let us add a result on spectral factorization of pseudocontinuable Schur functions.

Proposition 3.2. *Let* $f \in \mathcal{S}_{p\times q}(\mathbb{D}) \cap [\Pi(\mathbb{D})]^{p\times q}$, *let* $\rho := I_p - f\widehat{f^{\#}}$, *and let* $\sigma := I_q - \widehat{f^{\#}}f$. *Suppose that* $\det\rho$ *does not identically vanish in* $\mathbb{D}$. *Then* $\det\sigma$ *does not identically vanish in* $\mathbb{D}$ *and*

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log\det\underline{\rho}\,\mathrm{d}\underline{\lambda} = \frac{1}{2\pi}\int_{\mathbb{T}} \log\det\underline{\sigma}\,\mathrm{d}\underline{\lambda} > -\infty. \tag{2}$$

The unique normalized outer left spectral factor ϕ *of* $\langle\underline{\rho}\rangle$ *and the unique normalized outer right spectral factor* ψ *of* $\langle\underline{\sigma}\rangle$ *are functions which belong to* $\mathcal{S}_{p\times p}(\mathbb{D})$ *and* $\mathcal{S}_{q\times q}(\mathbb{D})$, *respectively, and which admit pseudocontinuations, namely*

$$\phi^{\#} = \hat{\rho}\widehat{\phi^{-1}} \quad \text{and} \quad \psi^{\#} = \widehat{\psi^{-1}}\hat{\sigma}. \tag{3}$$

Proof. Obviously, ρ belongs to $[\mathcal{NM}(\mathbb{D})]^{p\times p}$. In view of Remark 1.1, we have

$$\langle\det\underline{\rho}\rangle = \langle\det(I_p - \underline{f}\,\underline{f}^*)\rangle = \langle\det(I_q - \underline{f}^*\underline{f})\rangle = \langle\det\underline{\sigma}\rangle.$$

Since $\det\rho$ does not identically vanish in $\mathbb{D}$ we thus see that $\det\sigma$ does not identically vanish in $\mathbb{D}$ as well. Moreover, (2) follows. By assumption, we have

$$\langle I_p - \underline{f}\,\underline{f}^*\rangle = \langle\underline{\phi}\,\underline{\phi}^*\rangle \quad \text{and} \quad \langle I_q - \underline{f}^*\underline{f}\rangle = \langle\underline{\psi}^*\underline{\psi}\rangle. \tag{4}$$

Because $\phi \in \mathcal{S}_{p\times p}(\mathbb{D})$ and $\psi \in \mathcal{S}_{q\times q}(\mathbb{D})$ are outer functions, Remark 1.2 shows that $\phi^{-1} \in [\mathcal{N}_+(\mathbb{D})]^{p\times p} \subseteq [\mathcal{NM}(\mathbb{D})]^{p\times p}$ and $\psi^{-1} \in [\mathcal{N}_+(\mathbb{D})]^{q\times q} \subseteq [\mathcal{NM}(\mathbb{D})]^{q\times q}$. Consequently, $\widehat{\phi^{-1}} \in [\mathcal{NM}(\mathbb{E})]^{p\times p}$ and $\widehat{\psi^{-1}} \in [\mathcal{NM}(\mathbb{E})]^{q\times q}$. Because ρ and σ are matrix-valued functions of the meromorphic Nevanlinna class we get $\hat{\rho} \in [\mathcal{NM}(\mathbb{E})]^{p\times p}$ and $\hat{\sigma} \in [\mathcal{NM}(\mathbb{E})]^{q\times q}$. The fact that $\mathcal{NM}(\mathbb{E})$ is an algebra over $\mathbb{C}$ implies

$$\hat{\rho}\widehat{\phi^{-1}} \in [\mathcal{NM}(\mathbb{E})]^{p\times p} \quad \text{and} \quad \widehat{\psi^{-1}}\hat{\sigma} \in [\mathcal{NM}(\mathbb{E})]^{q\times q}. \tag{5}$$

Considering ρ and σ, Remark 1.1 and (4) we infer

$$\begin{aligned} \langle\underline{\hat{\rho}}\,\underline{\widehat{\phi^{-1}}}\rangle &= \langle\underline{\rho}^*(\underline{\phi^{-1}})^*\rangle = \langle\underline{\rho}(\underline{\phi}^*)^{-1}\rangle = \langle(I_p - \underline{f}\,\underline{f}^*)(\underline{\phi}^*)^{-1}\rangle \\ &= \langle(\underline{\phi}\,\underline{\phi}^*)(\underline{\phi}^*)^{-1}\rangle = \langle\underline{\phi}\rangle \end{aligned} \tag{6}$$

and analogously

$$\langle\underline{\widehat{\psi^{-1}}}\,\underline{\hat{\sigma}}\rangle = \langle\underline{\psi}\rangle. \tag{7}$$

From (5)–(7) it follows immediately that ϕ and ψ admit pseudocontinuations $\phi^{\#}$ and $\psi^{\#}$, respectively, which satisfy (3). □

4. A completion problem for matrix-valued inner functions

In this section, we will treat a completion problem for matrix-valued inner functions. Such completion problems were already introduced in [13, 1, 4]. Arov investigated this problem in context of system theory. The following considerations are based on the fact that inner functions always admit a pseudocontinuation. If U is a $(p+q)\times(p+q)$ matrix-valued function, the we will use the block partition

$$U=\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \tag{8}$$

where U_{11} is a $p\times p$ block. The completion problems we will discuss now are the following:

Problem CP1. Let $f\in\mathscr{S}_{p\times q}(\mathbb{D})$. Describe the set $\mathscr{I}_{CP1}(f)$ of all $(p+q)\times(p+q)$ matrix-valued inner functions U such that $U_{12}=f$. In particular, characterize the case that $\mathscr{I}_{CP1}(f)$ is nonempty.

Problem CP2. Let $g\in\mathscr{S}_{q\times p}(\mathbb{D})$. Describe the set $\mathscr{I}_{CP2}(g)$ of all $(p+q)\times(p+q)$ matrix-valued inner functions U such that $U_{21}=g$. In particular, characterize the case that $\mathscr{I}_{CP2}(g)$ is nonempty.

Remark 4.1. From Remark 2.2 it follows immediately that $\mathscr{I}_{CP1}(f)=\emptyset$ (respectively, $\mathscr{I}_{CP2}(g)=\emptyset$) if f (respectively, g) does not admit a pseudocontinuation.

To study these problems we will make an analysis of the block structure of matrix-valued inner functions.

Proposition 4.2. *Let U be an inner function that belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ with block partition* (8) *where U_{11} is a $p\times p$ block. Further, let*

$$\rho_1:=I_p-U_{12}\widehat{U_{12}^{\#}} \quad \textit{and} \quad \rho_2:=I_q-U_{21}\widehat{U_{21}^{\#}}, \tag{9}$$

and let

$$\sigma_1:=I_p-\widehat{U_{21}^{\#}}U_{21} \quad \textit{and} \quad \sigma_2:=I_q-\widehat{U_{12}^{\#}}U_{12}. \tag{10}$$

Then $\rho_1,\sigma_1\in[\mathscr{NM}(\mathbb{D})]^{p\times p}$ and $\rho_2,\sigma_2\in[\mathscr{NM}(\mathbb{D})]^{q\times q}$. If

$$\frac{1}{2\pi}\int_{\mathbb{T}}\log\det(I_p-\underline{U_{12}}\,\underline{U_{12}^{*}})\,\mathrm{d}\underline{\lambda}>-\infty, \tag{11}$$

then $\det\rho_1$ *and* $\det\rho_2$ *as well as* $\det\sigma_1$ *and* $\det\sigma_2$ *do not identically vanish in* $\mathbb{D}$.

Proof. Obviously, $\rho_1,\sigma_1\in[\mathscr{NM}(\mathbb{D})]^{p\times p}$ and $\rho_2,\sigma_2\in[\mathscr{NM}(\mathbb{D})]^{q\times q}$. Because U is an inner function the matrix $\underline{U}(z)$ is unitary for $\underline{\lambda}$-almost all $z\in\mathbb{T}$. Hence, the identities

$$I_p-\underline{U_{12}}\,\underline{U_{12}^{*}}=\underline{U_{11}}\,\underline{U_{11}^{*}} \quad \text{and} \quad I_p-\underline{U_{21}^{*}}\,\underline{U_{21}}=\underline{U_{11}^{*}}\,\underline{U_{11}} \tag{12}$$

and consequently

$$\det(I_p-\underline{U_{12}}\,\underline{U_{12}^{*}})=\det\underline{U_{11}}\,\det\underline{U_{11}^{*}}=\det(I_p-\underline{U_{21}^{*}}\,\underline{U_{21}}) \tag{13}$$

hold true $\underline{\lambda}$-a.e. on $\mathbb{T}$. If U_{12} satisfies (11), then the matrix $I_p-\underline{U_{12}}(z)\underline{U_{12}^{*}}(z)$ is positive Hermitian for $\underline{\lambda}$-almost all $z\in\mathbb{T}$. From (12) and (13) then we see that $I_p-\underline{U_{21}^{*}}(z)\underline{U_{21}}(z)$ is also positive Hermitian for $\underline{\lambda}$-almost all $z\in\mathbb{T}$. Thus, U_{12} and U_{21} have full Schur rank. That is why the functions $\det\rho_1$ and $\det\sigma_1$ do not identically vanish in $\mathbb{D}$. The other part of the assertion can be proved analogously. □

Proposition 4.3. *Let U be an inner function which belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ with block partition via* (8) *such that the $p\times q$ block U_{12} satisfies* (11). *Further, let ρ_1 and ρ_2 as well as σ_1 and σ_2 be defined by* (9) *and* (10). *Then the following statements hold true*:

(a) *There exist unique inner functions b_1, $c_1\in\mathscr{S}_{p\times p}(\mathbb{D})$ and b_2, $c_2\in\mathscr{S}_{q\times q}(\mathbb{D})$ and unique normalized outer functions $\phi_1,\psi_1\in\mathscr{S}_{p\times p}(\mathbb{D})$ and $\phi_2,\psi_2\in\mathscr{S}_{q\times q}(\mathbb{D})$ such that $U_{11}=\phi_1 b_1$, $U_{11}=c_1\psi_1$, $U_{22}=\phi_2 c_2$ and $U_{22}=b_2\psi_2$ hold true.*

(b) *The function ϕ_1 (respectively, ϕ_2) is the unique normalized outer left spectral factor of $\langle\underline{\rho_1}\rangle$ (respectively, $\langle\underline{\rho_2}\rangle$).*

(c) *The function ψ_1 (respectively, ψ_2) is the unique normalized outer right spectral factor of $\langle\underline{\sigma_1}\rangle$ (respectively, $\langle\underline{\sigma_2}\rangle$).*

(d) *The functions ϕ_1, ψ_1, ϕ_2 und ψ_2 admit pseudocontinuations $\phi_1^\#$, $\psi_1^\#$, $\phi_2^\#$ and $\psi_2^\#$, respectively, namely $\phi_1^\#=\widehat{\rho_1}(\widehat{\phi_1^{-1}})$, $\psi_1^\#=(\widehat{\psi_1^{-1}})\widehat{\sigma_1}$, $\phi_2^\#=\widehat{\rho_2}(\widehat{\phi_2^{-1}})$ and $\psi_2^\#=(\widehat{\psi_2^{-1}})\widehat{\sigma_2}$.*

Proof. Part (a) is a direct consequence of Theorem 3.1. The equalities

$$\langle\underline{\phi_1}\,\underline{\phi_1^*}\rangle=\langle\underline{U_{11}}\,\underline{U_{11}^*}\rangle=\langle I_p-\underline{U_{12}}\,\underline{U_{12}^*}\rangle=\langle\underline{\rho_1}\rangle \tag{14}$$

and

$$\langle\underline{\psi_1^*}\,\underline{\psi_1}\rangle=\langle\underline{U_{11}^*}\,\underline{U_{11}}\rangle=\langle I_p-\underline{U_{21}^*}\,\underline{U_{21}}\rangle=\langle\underline{\sigma_1}\rangle \tag{15}$$

are fulfilled because b_1 and c_1 are inner functions and U is an inner function. Hence, ϕ_1 is the unique normalized outer left spectral factor of $\langle\underline{\rho_1}\rangle$ and ψ_1 is the unique normalized outer right spectral factor of $\langle\underline{\sigma_1}\rangle$. The other statements in parts (b) and (c) can be obtained similarly. Since ϕ_1 is an outer function in $\mathscr{S}_{p\times p}(\mathbb{D})$ we get from Remark 1.2 that $\phi_1^{-1}\in[\mathscr{N}_+(\mathbb{D})]^{p\times p}\subseteq[\mathscr{NM}(\mathbb{D})]^{p\times p}$. Therefore, $\widehat{\phi_1^{-1}}\in[\mathscr{NM}(\mathbb{E})]^{p\times p}$. From Proposition 4.2 it follows $\widehat{\rho_1}\in[\mathscr{NM}(\mathbb{E})]^{p\times p}$. From the fact that $\mathscr{NM}(\mathbb{E})$ is an algebra over $\mathbb{C}$ we can conclude that

$$\widehat{\rho_1}(\widehat{\phi_1^{-1}})\in[\mathscr{NM}(\mathbb{E})]^{p\times p}. \tag{16}$$

Applying (9) and Remark 1.1 we get

$$\begin{aligned}\langle\underline{\widehat{\rho_1}(\widehat{\phi_1^{-1}})}\rangle&=\langle\underline{\rho_1^*(\phi_1^{-1})^*}\rangle=\langle\underline{\rho_1}(\underline{\phi_1^{-1}})^*\rangle=\langle(I_p-\underline{U_{12}}\,\underline{U_{12}^*})(\underline{\phi_1^{-1}})^*\rangle\\&=\langle\underline{\phi_1}\,\underline{\phi_1^*}(\underline{\phi_1^*})^{-1}\rangle=\langle\underline{\phi_1}\rangle.\end{aligned} \tag{17}$$

With (16) and (17) we see that $\widehat{\rho_1}(\widehat{\phi_1^{-1}})$ is a pseudocontinuation of ϕ_1, i.e., $\phi_1^\#=\widehat{\rho_1}(\widehat{\phi_1^{-1}})$ holds true. Using the fact that ψ_1 is an outer function in $\mathscr{S}_{p\times p}(\mathbb{D})$ and Remark 1.2 it follows that $\psi_1^{-1}\in[\mathscr{N}_+(\mathbb{D})]^{p\times p}\subseteq[\mathscr{NM}(\mathbb{D})]^{p\times p}$. Hence, $\widehat{\psi_1^{-1}}\in[\mathscr{NM}(\mathbb{E})]^{p\times p}$. Proposition 4.2 yields $\widehat{\sigma_1}\in[\mathscr{NM}(\mathbb{E})]^{p\times p}$. From the fact that $\mathscr{NM}(\mathbb{E})$ is an algebra over $\mathbb{C}$ it follows then

$$(\widehat{\psi_1^{-1}})\widehat{\sigma_1}\in[\mathscr{NM}(\mathbb{E})]^{p\times p}. \tag{18}$$

Thus, Remark 1.1 provides

$$\begin{aligned}\langle\underline{(\widehat{\psi_1^{-1}})\widehat{\sigma_1}}\rangle&=\langle\underline{(\psi_1^{-1})^*\sigma_1^*}\rangle=\langle(\underline{\psi_1^{-1}})^*\underline{\sigma_1}\rangle=\langle(\underline{\psi_1^{-1}})^*(I_p-\underline{U_{21}^*}\,\underline{U_{21}})\rangle\\&=\langle(\underline{\psi_1^*})^{-1}\underline{\psi_1^*}\,\underline{\psi_1}\rangle=\langle\underline{\psi_1}\rangle.\end{aligned} \tag{19}$$

In view of (18) and (19), we see that $\psi_1^{\#} = \widehat{(\psi_1^{-1})}\widehat{\sigma_1}$. The rest of the assertion can be proved analogously. □

Proposition 4.4. *Let U be an inner function that belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ with block partition (8) such that the $p \times q$ block U_{12} satisfies (11). Let the functions ρ_1, ρ_2, σ_1 and σ_2 be defined by (9) and (10). Further, let the inner functions b_1, b_2, c_1, c_2 and the normalized outer functions $\phi_1, \phi_2, \psi_1, \psi_2$ be choosen like in Proposition 4.3. Then U admits the representations*

$$U = \operatorname{diag}(I_p, b_2)\begin{pmatrix} \phi_1 & U_{12} \\ -\psi_2 \widehat{U_{12}^{\#}\rho_1^{-1}}\phi_1 & \psi_2 \end{pmatrix}\operatorname{diag}(b_1, I_q) \tag{20}$$

and

$$U = \operatorname{diag}(c_1, I_q)\begin{pmatrix} \psi_1 & -\psi_1\widehat{U_{21}^{\#}\rho_2^{-1}}\phi_2 \\ U_{21} & \phi_2 \end{pmatrix}\operatorname{diag}(I_p, c_2). \tag{21}$$

Moreover,

$$b_2\psi_2\widehat{U_{12}^{\#}\rho_1^{-1}}\phi_1 b_1 = -U_{21} \in \mathscr{S}_{q\times p}(\mathbb{D}) \tag{22}$$

and

$$c_1\psi_1\widehat{U_{21}^{\#}\rho_2^{-1}}\phi_2 c_2 = -U_{12} \in \mathscr{S}_{p\times q}(\mathbb{D}). \tag{23}$$

Proof. According to $\phi_1 b_1 = U_{11}$ and $b_2\psi_2 = U_{22}$ it is sufficient to show that (22) holds true in order to verify (20). In view of Proposition 4.2, we have

$$-b_2\psi_2\widehat{U_{12}^{\#}\rho_1^{-1}}\phi_1 b_1 = -U_{22}\widehat{U_{12}^{\#}\rho_1^{-1}}U_{11}. \tag{24}$$

Since U is an inner function satisfying (11) we have

$$\langle I_p - \underline{U_{12}}\,\underline{U_{12}^*}\rangle = \langle \underline{U_{11}}\,\underline{U_{11}^*}\rangle \quad \text{and} \quad \langle \underline{U_{21}}\rangle = \langle -\underline{U_{22}}\,\underline{U_{12}^*}(\underline{U_{11}^*})^{-1}\rangle. \tag{25}$$

Consequently, Remark 1.1 yields

$$\begin{aligned}\langle -\underline{U_{22}}\,\underline{\widehat{U_{12}^{\#}}}\,\underline{\rho_1^{-1}}\,\underline{U_{11}}\rangle &= \langle -\underline{U_{22}}\,\underline{U_{12}^*}(I_p - \underline{U_{12}}\,\underline{U_{12}^*})^{-1}\underline{U_{11}}\rangle \\ &= \langle -\underline{U_{22}}\,\underline{U_{12}^*}(\underline{U_{11}}\,\underline{U_{11}^*})^{-1}\underline{U_{11}}\rangle \\ &= \langle -\underline{U_{22}}\,\underline{U_{12}^*}(\underline{U_{11}^*})^{-1}\rangle = \langle \underline{U_{21}}\rangle. \end{aligned} \tag{26}$$

Since each entry of the matrix-valued functions $-U_{22}$, $\widehat{U_{12}^{\#}}$, ρ_1^{-1} and U_{11} belongs to $\mathscr{N}\mathscr{M}(\mathbb{D})$, the fact that $\mathscr{N}\mathscr{M}(\mathbb{D})$ is an algebra over $\mathbb{C}$ implies that

$$-U_{22}\widehat{U_{12}^{\#}\rho_1^{-1}}U_{11} \in [\mathscr{N}\mathscr{M}(\mathbb{D})]^{q\times p}. \tag{27}$$

Because an arbitrary function which belongs to $[\mathscr{N}\mathscr{M}(\mathbb{D})]^{q\times p}$ is uniquely determined by its boundary values on the unit circle, we see from (26) and (27) that

$$-U_{22}\widehat{U_{12}^{\#}\rho_1^{-1}}U_{11} = U_{21}. \tag{28}$$

Comparing (24) and (28) we obtain (22). Thus, (20) is verified. The relations (21) and (23) can be proved analogously. □

Proposition 4.5. *Let U be an inner function which belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ with block partition via* (8) *such that the $p\times q$ block U_{12} satisfies* (11). *Let ρ_1 and ρ_2 as well as σ_1 and σ_2 be defined by* (9) *and* (10). *Further, let the inner functions b_1, b_2, c_1, c_2 and the normalized outer functions $\phi_1, \phi_2, \psi_1, \psi_2$ be choosen like in Proposition* 4.3. *If U_{11} and U_{22} are matrix-valued outer functions, then the functions b_1 and b_2 as well as c_1 and c_2 are constant inner functions with unitary values and the relations*

$$\psi_2 \widehat{U_{12}^{\#}} \rho_1^{-1} \phi_1 \in \mathscr{S}_{q\times p}(\mathbb{D}), \qquad \psi_1 \widehat{U_{21}^{\#}} \rho_2^{-1} \phi_2 \in \mathscr{S}_{p\times q}(\mathbb{D}), \tag{29}$$

$$\widehat{U_{12}^{\#}} \rho_1^{-1} \in [\mathscr{N}_+(\mathbb{D})]^{q\times p} \quad \text{and} \quad \widehat{U_{21}^{\#}} \rho_2^{-1} \in [\mathscr{N}_+(\mathbb{D})]^{p\times q} \tag{30}$$

are fulfilled.

Proof. Since U belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ we have $U_{11} \in \mathscr{S}_{p\times p}(\mathbb{D}) \subseteq [H^2(\mathbb{D})]^{p\times p}$ and $U_{22} \in \mathscr{S}_{q\times q}(\mathbb{D}) \subseteq [H^2(\mathbb{D})]^{q\times q}$. In particular, $\Lambda_1 := \underline{U_{11}}\,\underline{U_{11}}^*$, $\Lambda_2 := \underline{U_{22}}\,\underline{U_{22}}^*$, $\Xi_1 := \underline{U_{11}}^*\underline{U_{11}}$ and $\Xi_2 := \underline{U_{22}}^*\underline{U_{22}}$ are Lebesgue integrable functions on $\mathbb{T}$ with nonnegative Hermitian values. Obviously, ϕ_1 (respectively, ϕ_2) is the unique normalized outer left spectral factor of $\langle \Lambda_1 \rangle$ (respectively, $\langle \Lambda_2 \rangle$), and ψ_1 (respectively, ψ_2) is the unique normalized outer right spectral factor of $\langle \Xi_1 \rangle$ (respectively, $\langle \Xi_2 \rangle$). Suppose that U_{11} and U_{22} are outer functions. Then we see that U_{11} (respectively, U_{22}) is as well an outer left spectral factor of $\langle \Lambda_1 \rangle$ (respectively, $\langle \Lambda_2 \rangle$) as an outer right spectral factor of $\langle \Xi_1 \rangle$ (respectively, $\langle \Xi_2 \rangle$). Thus, we obtain from Theorem 3.1 that the inner functions b_1 and c_1 as well as b_2 and c_2 are constant functions in $\mathbb{D}$ with some unitary values $u_1, v_1 \in \mathbb{C}^{p\times p}$ and $u_2, v_2 \in \mathbb{C}^{q\times q}$, respectively. Then Eq. (29) follows immediately from [14, Remark 1.1.2] and Proposition 4.4. Since $\phi_1, \psi_1 \in \mathscr{S}_{p\times p}(\mathbb{D})$ and $\phi_2, \psi_2 \in \mathscr{S}_{q\times q}(\mathbb{D})$ are outer functions, Remark 1.2 implies that ϕ_1^{-1} and ψ_1^{-1} as well as ϕ_2^{-1} and ψ_2^{-1} are outer functions of $[\mathscr{N}_+(\mathbb{D})]^{p\times p}$ and $[\mathscr{N}_+(\mathbb{D})]^{q\times q}$, respectively. Because of $\mathscr{S}_{p\times q}(\mathbb{D}) \subseteq [\mathscr{N}_+(\mathbb{D})]^{p\times q}$, $\mathscr{S}_{q\times p}(\mathbb{D}) \subseteq [\mathscr{N}_+(\mathbb{D})]^{q\times p}$ and the fact that $\mathscr{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$, (30) follows immediately. □

Proposition 4.6. *Let $f \in \mathscr{S}_{p\times q}(\mathbb{D}) \cap [\Pi(\mathbb{D})]^{p\times q}$, let $\rho_1 := I_p - f\widehat{f^{\#}}$, and let $\sigma_2 := I_q - \widehat{f^{\#}} f$. Suppose that $\det \rho_1$ does not identically vanish in $\mathbb{D}$. Let ϕ_1 and ψ_2 be the unique normalized outer left spectral factor of $\langle \rho_1 \rangle$ and the unique normalized outer right spectral factor of $\langle \sigma_2 \rangle$, respectively. Further, let $b_1 \in \mathscr{S}_{p\times p}(\mathbb{D})$ and $b_2 \in \mathscr{S}_{q\times q}(\mathbb{D})$ be inner functions such that*

$$b_2 \psi_2 \widehat{f^{\#}} \rho_1^{-1} \phi_1 b_1 \in [\mathscr{N}_+(\mathbb{D})]^{q\times p}. \tag{31}$$

Then

$$U := \operatorname{diag}(I_p, b_2) \begin{pmatrix} \phi_1 & f \\ -\psi_2 \widehat{f^{\#}} \rho_1^{-1} \phi_1 & \psi_2 \end{pmatrix} \operatorname{diag}(b_1, I_q) \tag{32}$$

is an inner function which belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$.

Proof. First, we observe that Proposition 3.2 yields

$$\frac{1}{2\pi}\int_{\mathbb{T}} \log\det \underline{\rho_1}\, \mathrm{d}\underline{\lambda} = \frac{1}{2\pi}\int_{\mathbb{T}} \log\det \underline{\sigma_2}\, \mathrm{d}\underline{\lambda} > -\infty.$$

Let (8) be the block partition of U with $p\times p$ block U_{11}. Using (31), (32), $H^2(\mathbb{D})\subseteq \mathcal{N}_+(\mathbb{D})$, the facts that each entry of a matricial Schur function defined on $\mathbb{D}$ belongs to $\mathcal{N}_+(\mathbb{D})$ and that $\mathcal{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$, we can conclude

$$U_{11} = \phi_1 b_1 \in [\mathcal{N}_+(\mathbb{D})]^{p\times p}, \qquad U_{12} = f \in [\mathcal{N}_+(\mathbb{D})]^{p\times q},$$

$$U_{21} = -b_2\psi_2 \widehat{f^{\#}} \rho_1^{-1}\phi_1 b_1 \in [\mathcal{N}_+(\mathbb{D})]^{q\times p}, \qquad U_{22} = b_2\psi_2 \in [\mathcal{N}_+(\mathbb{D})]^{q\times q}$$

and consequently

$$U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}. \tag{33}$$

In view of (32), Remark 1.1 and $\langle \underline{b_1}\,\underline{b_1^*}\rangle = \langle I_p\rangle$, we infer

$$\langle \underline{U}\,\underline{U^*}\rangle = \left\langle\!\!\left\langle \begin{pmatrix} \underline{\phi_1}\,\underline{\phi_1^*}+\underline{f}\,\underline{f^*} & -\underline{\phi_1}\,\underline{\phi_1^*}\,\underline{\rho_1^{-1}}\,\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*}+\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*} \\ -\underline{b_2}\,\underline{\psi_2}\,\underline{f^*}\,\underline{\rho_1^{-1}}\,\underline{\phi_1}\,\underline{\phi_1^*}+\underline{b_2}\,\underline{\psi_2}\,\underline{f^*} & \underline{b_2}\,\underline{\psi_2}\,\underline{f^*}\,\underline{\rho_1^{-1}}\,\underline{\phi_1}\,\underline{\phi_1^*}\,\underline{\rho_1^{-1}}\,\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*}+\underline{b_2}\,\underline{\psi_2}\,\underline{\psi_2^*}\,\underline{b_2^*} \end{pmatrix}\right\rangle\!\!\right\rangle. \tag{34}$$

Since ϕ_1 is a left spectral factor of $\langle\underline{\rho_1}\rangle$, we get from Remark 1.1 that

$$\langle \underline{\phi_1}\,\underline{\phi_1^*} + \underline{f}\,\underline{f^*}\rangle = \langle I_p\rangle \tag{35}$$

and

$$\begin{aligned}\langle -\underline{\phi_1}\,\underline{\phi_1^*}\,\underline{\rho_1^{-1}}\,\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*} + \underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*}\rangle &= \langle [-\underline{\phi_1}\,\underline{\phi_1^*}\,\underline{\rho_1^{-1}} + I_p]\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*}\rangle \\ &= \langle 0_{p\times p}\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*}\rangle = \langle 0_{p\times q}\rangle. \end{aligned}\tag{36}$$

Since b_2 is an inner function, ϕ_1 is a left spectral factor of $\langle\underline{\rho_1}\rangle$ and ψ_2 is a right spectral factor of $\langle I_q - \underline{f^*}\,\underline{f}\rangle$, it follows

$$\begin{aligned}&\langle \underline{b_2}\underline{\psi_2}\underline{f^*}\,\underline{\rho_1^{-1}}\,\underline{\phi_1}\,\underline{\phi_1^*}\,\underline{\rho_1^{-1}}\,\underline{f}\,\underline{\psi_2^*}\,\underline{b_2^*} + \underline{b_2}\underline{\psi_2}\;\underline{\psi_2^*}\,\underline{b_2^*}\rangle = \langle \underline{b_2}\underline{\psi_2}\underline{f^*}\,\underline{\rho_1^{-1}}\,\underline{f}\,\underline{\psi_2^*}\;\underline{b_2^*} + \underline{b_2}\underline{\psi_2}\underline{\psi_2^*}\,\underline{b_2^*}\rangle \\ &= \langle \underline{b_2}\underline{\psi_2}[\underline{f^*}\,\underline{\rho_1^{-1}}\,\underline{f} + I_q]\underline{\psi_2^*}\,\underline{b_2^*}\rangle = \langle \underline{b_2}\underline{\psi_2}[\underline{f^*}\,\underline{f}(I_q - \underline{f^*}\,\underline{f})^{-1} + I_q]\underline{\psi_2^*}\,\underline{b_2^*}\rangle \\ &= \langle \underline{b_2}\underline{\psi_2}(I_q - \underline{f^*}\,\underline{f})^{-1}\underline{\psi_2^*}\,\underline{b_2^*}\rangle = \langle \underline{b_2}\underline{\psi_2}(\underline{\psi_2^*}\,\underline{\psi_2})^{-1}\underline{\psi_2^*}\,\underline{b_2^*}\rangle = \langle \underline{b_2}\,\underline{b_2^*}\rangle \\ &= \langle I_q\rangle. \end{aligned}\tag{37}$$

Combining (34)–(37) we get $\langle \underline{U}\,\underline{U^*}\rangle = \langle I_{p+q}\rangle$. This fact, (33) and the maximum modulus principle for the Smirnov class (see, e.g., [16, Theorem 2.11]) provide that U is an inner function which belongs to the Schur class $\mathcal{S}_{(p+q)\times(p+q)}(\mathbb{D})$. $\square$

Proposition 4.7. *Let $g\in\mathcal{S}_{q\times p}(\mathbb{D})\cap[\Pi(\mathbb{D})]^{q\times p}$, let $\rho_2 := I_p - \widehat{gg^{\#}}$, and let $\sigma_1 := I_q - \widehat{g^{\#}}g$. Suppose that $\det\rho_2$ does not identically vanish in $\mathbb{D}$. Let ϕ_2 and ψ_1 be the unique normalized outer left* *spectral factor of $\langle\rho_2\rangle$ and the unique normalized outer right spectral factor of $\langle\sigma_1\rangle$, respectively. Further, let $c_1 \in \mathscr{S}_{p\times p}(\mathbb{D})$ and $c_2 \in \mathscr{S}_{q\times q}(\mathbb{D})$ be inner functions such that*

$$c_1\psi_1\widehat{g^{\#}}\rho_2^{-1}\phi_2c_2 \in [\mathscr{N}_+(\mathbb{D})]^{p\times q} \;. \tag{38}$$

Then

$$V := \operatorname{diag}(c_1, I_q)\begin{pmatrix} \psi_1 & -\psi_1\widehat{g^{\#}}\rho_2^{-1}\phi_2 \\ g & \phi_2 \end{pmatrix}\operatorname{diag}(I_p, c_2) \tag{39}$$

is an inner function which belongs to $\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$.

Proof. Proposition 4.7 can be analogously proved as Proposition 4.6. We omit the details. □

Now we are able to give an answer to Problem CP1.

Theorem 4.8. *Let $f \in \mathscr{S}_{p\times q}(\mathbb{D})$ be a pseudocontinuable function such that $\det(I_p - f\widehat{f^{\#}})$ does not identically vanish in $\mathbb{D}$. Let $\rho_1 := I_p - f\widehat{f^{\#}}$, and let $\sigma_2 := I_q - \widehat{f^{\#}}f$. Further, let ϕ_1 be the unique normalized outer left spectral factor of $\langle\underline{\rho_1}\rangle$, and let ψ_2 be the unique normalized outer right spectral factor of $\langle\underline{\sigma_2}\rangle$. Then the following statements hold true:*

(a) *There exists an inner function $U \in \mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ such that the right upper $p\times q$ block U_{12} of U coincides with f.*

(b) *Let U be a function which belongs to $[\mathscr{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ with block partition via (8), where U_{12} is a $p\times q$ block. Then the following statements are equivalent:*

- (i) *U is an inner function which satisfies $U_{12} = f$.*
- (ii) *There exist inner functions $b_1 \in \mathscr{S}_{p\times p}(\mathbb{D})$ and $b_2 \in \mathscr{S}_{q\times q}(\mathbb{D})$ such that $b_2\psi_2\widehat{f^{\#}}\rho_1^{-1}\phi_1b_1$ belongs to $[\mathscr{N}_+(\mathbb{D})]^{q\times p}$ and that U admits the representation*

$$U = \operatorname{diag}(I_p, b_2)\begin{pmatrix} \phi_1 & f \\ -\psi_2\widehat{f^{\#}}\rho_1^{-1}\phi_1 & \psi_2 \end{pmatrix}\operatorname{diag}(b_1, I_q). \tag{40}$$

Proof. In view of Proposition 4.3, the application of Propositions 4.4 and 4.6 yield the assertion. □

The following theorem which also follows immediately from Propositions 4.4, 4.6 and 4.7 gives an answer to Problem CP2.

Theorem 4.9. *Let $g \in \mathscr{S}_{q\times p}(\mathbb{D})$ be a pseudocontinuable function such that $\det(I_p - g\widehat{g^{\#}})$ does not identically vanish in $\mathbb{D}$. Let $\rho_2 := I_q - g\widehat{g^{\#}}$, and let $\sigma_1 := I_p - \widehat{g^{\#}}g$. Further, let ϕ_2 be the unique normalized outer left spectral factor of $\langle\underline{\rho_2}\rangle$, and let ψ_1 be the unique normalized outer right spectral factor of $\langle\underline{\sigma_1}\rangle$. Then the following statements hold true:*

(a) *There exists an inner function $U \in \mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ such that the left lower $q\times p$ block U_{21} of U coincides with g.*

(b) *Let U be a function which belongs to $[\mathscr{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ with block partition via (8), where U_{21} is a $q\times p$ block. Then the following statements are equivalent:*

- (i) *U is an inner function which satisfies $U_{21} = g$.*

(ii) *There exist inner functions* $c_1 \in \mathscr{S}_{p\times p}$ *and* $c_2 \in \mathscr{S}_{q\times q}(\mathbb{D})$ *such that* $c_1\psi_1\widehat{g^{\#}}\rho_2^{-1}\phi_2 c_2$ *belongs to* $[\mathscr{N}_+(\mathbb{D})]^{p\times q}$ *and that* U *admits the representation*

$$U = \operatorname{diag}(c_1, I_q)\begin{pmatrix} \psi_1 & -\psi_1\widehat{g^{\#}}\rho_2^{-1}\phi_2 \\ g & \phi_2 \end{pmatrix}\operatorname{diag}(I_p, c_2). \tag{41}$$

It should be mentioned that the results of this section can be obtained in an alternate way with Arovs results in context of the description of the set of all Darlington realizations of pseudocontinuable Schur functions.

5. Completion problems for subclasses of matrix-valued inner functions

In this section, we will discuss completion problems for particular matrix-valued inner functions. The subclasses we will consider play an essential role in the context of a Nehari-type interpolation problem (see [15]) and are associated with certain subclasses of j_{pq}-inner functions via Potapov–Ginzburg transform (see [8–10])

If a $(p+q)\times(p+q)$ matrix-valued function U is given, then we will continue to use the block partition (8), where U_{11} is a $p\times p$ block. An inner function $U \in \mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ is said to be *of* (p,q)*-Smirnov type* (respectively, *of inverse* (p,q)*-Smirnov type*) if U_{22} is an outer function in $\mathscr{S}_{q\times q}(\mathbb{D})$ (respectively, if U_{11} is an outer function in $\mathscr{S}_{p\times p}(\mathbb{D})$). An inner function $U\in\mathscr{S}_{(p+q)\times(p+q)}(\mathbb{D})$ is said to be a (p,q)*-type Arov-inner* function if both functions U_{11} and U_{22} are outer matrix-valued Schur functions.

If $\mathfrak{M}$ is one of the subclasses of inner $(p+q)\times(p+q)$ Schur functions introduced above, then we are going to study the following two types of completion problems.

Problem CP3. Let $f\in\mathscr{S}_{p\times q}(\mathbb{D})$. Describe the set of all $U\in\mathfrak{M}$ such that $U_{12}=f$.

Problem CP4. Let $g\in\mathscr{S}_{q\times p}(\mathbb{D})$. Describe the set of all $U\in\mathfrak{M}$ such that $U_{21}=g$.

Theorem 5.1. *Let* $f\in\mathscr{S}_{p\times q}(\mathbb{D})\cap[\Pi(\mathbb{D})]^{p\times q}$, *let* $\varrho_1 := I_p - f\widehat{f^{\#}}$, *and let* $\sigma_2 := I_q - \widehat{f^{\#}}f$. *Suppose that* $\det\varrho_1$ *does not identically vanish in* $\mathbb{D}$. *Let* Φ_1 *and* Ψ_2 *be the unique normalized outer left spectral factor of* $\langle\underline{\varrho_1}\rangle$ *and the unique normalized outer right spectral factor of* $\langle\underline{\sigma_2}\rangle$. *Then there exists an inner function* U *of* (p,q)*-Smirnov type which satisfies* $U_{12}=f$ *if and only if there is an inner function* $b_1\in\mathscr{S}_{p\times p}(\mathbb{D})$ *such that*

$$\widehat{f^{\#}}\varrho_1^{-1}\Phi_1 b_1\in[\mathscr{N}_+(\mathbb{D})]^{q\times p} \tag{42}$$

holds true. In this case, a given function $U\in[\mathscr{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ *is an inner function of* (p,q)*-Smirnov type which fulfills* $U_{12}=f$ *if and only if there are an inner function* $b_1\in\mathscr{S}_{p\times p}(\mathbb{D})$ *satisfying* (42) *and a* $q\times q$ *unitary complex matrix* u_2 *such that*

$$U = \operatorname{diag}(I_p, u_2)\begin{pmatrix} \Phi_1 & f \\ -\Psi_2\widehat{f^{\#}}\varrho_1^{-1}\Phi_1 & \Psi_2 \end{pmatrix}\operatorname{diag}(b_1, I_q). \tag{43}$$

Proof. First, assume that b_1 is an inner function which belongs to $\mathscr{S}_{p\times p}(\mathbb{D})$ and which satisfies (42). For each $q\times q$ unitary matrix u_2, then $u_2\Psi_2\widehat{f^{\#}\varrho_1^{-1}\Phi_1}b_1\in[\mathscr{N}_+(\mathbb{D})]^{q\times p}$ follows, and Proposition 4.6 and Remark 1.2 yield that the matrix-valued function U which admits the representation (43) is an inner function of (p,q)-Smirnov type with $U_{12}=f$. Conversely, if there is an inner function U of (p,q)-Smirnov type which satisfies $U_{12}=f$, then Theorem 4.8 shows the existence of inner functions $b_1\in\mathscr{S}_{p\times p}(\mathbb{D})$ and $b_2\in\mathscr{S}_{q\times q}(\mathbb{D})$ such that

$$b_2\Psi_2\widehat{f^{\#}}\varrho_1^{-1}\Phi_1 b_1\in[\mathscr{N}_+(\mathbb{D})]^{q\times p} \tag{44}$$

and that U admits the representation (40). In particular, we thus see that the outer $q\times q$ Schur function U_{22} admits the representation $U_{22}=b_2\Psi_2$. Hence, U_{22} is a right spectral factor of $\langle\underline{\sigma_2}\rangle$. Since $\det\varrho_1$ does not identically vanish, we infer that $\det\sigma_2$ does not identically vanish (see, e.g., [14, Lemma 1.1.12]). Thus, Proposition 3.2, Theorem 3.1 and Remark 1.2 provide that b_2 is a constant function the value of which is some unitary matrix $u_2\in\mathbb{C}^{q\times q}$. Consequently, (43) follows from (40). Moreover, then (44) and the fact that $\mathscr{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$ imply (42). □

Theorem 5.2. *Let the assumptions of Theorem 5.1 be satisfied. Then there exists an inner function U of inverse (p,q)-Smirnov type which satisfies $U_{12}=f$ if and only if there is an inner function $b_2\in\mathscr{S}_{q\times q}(\mathbb{D})$ such that*

$$b_2\Psi_2\widehat{f^{\#}}\varrho_1^{-1}\in[\mathscr{N}_+(\mathbb{D})]^{q\times p} \tag{45}$$

holds true. In this case, a given function $U\in[\mathscr{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ is an inner function of inverse (p,q)-Smirnov type which fulfills $U_{12}=f$ if and only if there are an inner function $b_2\in\mathscr{S}_{q\times q}(\mathbb{D})$ satisfying (45) and a $p\times p$ unitary complex matrix u_1 such that

$$U=\operatorname{diag}(I_p,b_2)\begin{pmatrix}\Phi_1 & f\\ -\Psi_2\widehat{f^{\#}}\varrho_1^{-1}\Phi_1 & \Psi_2\end{pmatrix}\operatorname{diag}(u_1,I_q). \tag{46}$$

Proof. First, assume that b_2 is an inner function which belongs to $\mathscr{S}_{q\times q}(\mathbb{D})$ and which satisfies (45). For every $p\times p$ unitary matrix u_1, then $b_2\Psi_2\widehat{f^{\#}}\varrho_1^{-1}\Phi_1 u_1$ belongs to $[\mathscr{N}_+(\mathbb{D})]^{q\times p}$, and in view of Proposition 4.6 and Remark 1.2, the matrix-valued function U which admits the representation (46) is an inner function of inverse (p,q)-Smirnov type with $U_{12}=f$. Conversely, now assume that there exists an inner function U of inverse (p,q)-Smirnov type which satisfies $U_{12}=f$. Applying Theorem 4.8 then we see that there are inner functions $b_1\in\mathscr{S}_{p\times p}(\mathbb{D})$ and $b_2\in\mathscr{S}_{q\times q}(\mathbb{D})$ such that (44) is satisfied and that U admits the representation (40). In particular, $U_{11}=\Phi_1 b_1$. Consequently, U_{11} is a left spectral factor of $\langle\underline{\varrho_1}\rangle$. Since $\det\varrho_1$ does not identically vanish, we obtain from Proposition 3.2, Theorem 3.1 and Remark 1.2 that b_1 is a constant function the value of which is some $p\times p$ unitary matrix u_1. Therefore, (40) implies (46). Moreover, we get from (44) that (45) is fulfilled. □

Theorem 5.3. *Let the assumptions of Theorem 5.1 be satisfied. Then there exists a (p,q)-type Arov-inner function U which satisfies $U_{12}=f$ if and only if*

$$\widehat{f^{\#}}\varrho_1^{-1}\in[\mathscr{N}_+(\mathbb{D})]^{q\times p} \tag{47}$$

holds true. In this case, a given function $U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ *is a* (p,q)*-type Arov-inner function which satisfies* $U_{12} = f$ *if and only if there are a* $p\times p$ *unitary complex matrix* u_1 *and a* $q \times q$ *unitary complex matrix* u_2 *such that*

$$U = \operatorname{diag}(I_p, u_2)\begin{pmatrix} \Phi_1 & f \\ -\Psi_2\widehat{f^\#}\varrho_1^{-1}\Phi_1 & \Psi_2 \end{pmatrix}\operatorname{diag}(u_1, I_q). \tag{48}$$

Proof. If (47) is satisfied, then the fact that $\mathcal{N}_+(\mathbb{D})$ is an algebra over $\mathbb{C}$, Theorems 5.1 and 5.2 provide that, for all unitary matrices $u_1 \in \mathbb{C}^{p\times p}$ and $u_2 \in \mathbb{C}^{q\times q}$, the matrix-valued function $U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ given by (48) is an (p,q)-type Arov-inner function which fulfills $U_{12} = f$. Conversely, now suppose that there is a (p,q)-type Arov-inner function U which satisfies $U_{12} = f$. Then Theorem 5.1 shows that (42) is valid and that U admits the representation (43) with some inner function $b_1 \in \mathcal{S}_{p\times p}(\mathbb{D})$ and some $q \times q$ unitary matrix u_2. In particular, $U_{11} = \Phi_1 b_1$. As in the proof of Theorem 5.2 we get that b_1 is a constant function the value of which is a $p \times p$ unitary matrix u_1. In view of Remark 1.2, (47) and (48) follow. □

Finally, we will state the corresponding results if the left lower $q \times p$ block in the completion problems for the considered subclasses of inner matrix-valued functions is given. The proofs are analoguous to those of Theorems 5.1–5.3. We omit the details.

Theorem 5.4. *Let* $g \in \mathcal{S}_{q\times p}(\mathbb{D}) \cap [\Pi(\mathbb{D})]^{q\times p}$, *let* $\varrho_2 := I_q - \widehat{gg^\#}$, *and let* $\sigma_1 := I_p - \widehat{g^\# g}$. *Suppose that* $\det \varrho_2$ *does not identically vanish in* $\mathbb{D}$. *Let* Φ_2 *and* Ψ_1 *be the unique normalized outer left spectral factor of* $\langle \underline{\varrho_2} \rangle$ *and the unique normalized outer right spectral factor of* $\langle \underline{\sigma_1} \rangle$, *respectively. Then there exists an inner function* U *of* (p,q)*-Smirnov type which satisfies* $U_{21} = g$ *if and only if there is an inner function* $c_1 \in \mathcal{S}_{p\times p}(\mathbb{D})$ *such that*

$$c_1\Psi_2\widehat{g^\#}\varrho_2^{-1} \in [\mathcal{N}_+(\mathbb{D})]^{p\times q} \tag{49}$$

holds true. In this case, a given function $U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ *is a function of* (p,q)*-Smirnov type which fulfills* $U_{21} = g$ *if and only if there are an inner function* $c_1 \in \mathcal{S}_{p\times p}(\mathbb{D})$ *satisfying* (49) *and a* $q \times q$ *unitary complex matrix* u_2 *such that*

$$U = \operatorname{diag}(c_1, I_q)\begin{pmatrix} \Psi_1 & -\Psi_1\widehat{g^\#}\varrho_2^{-1}\Phi_2 \\ g & \Phi_2 \end{pmatrix}\operatorname{diag}(I_p, u_2). \tag{50}$$

Theorem 5.5. *Let the assumptions of Theorem 5.4 be satisfied. Then there exists an inner function* U *of inverse* (p,q)*-Smirnov type which satisfies* $U_{21} = g$ *if and only if there is an inner function* $c_2 \in \mathcal{S}_{q\times q}(\mathbb{D})$ *such that*

$$\widehat{g^\#}\varrho_2^{-1}\Phi_2 c_2 \in [\mathcal{N}_+(\mathbb{D})]^{p\times q} \tag{51}$$

holds true. In this case, a given function $U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ *is an inner function of inverse* (p,q)*-Smirnov type which fulfills* $U_{21} = g$ *if and only if there are an inner function* $c_2 \in \mathcal{S}_{q\times q}(\mathbb{D})$ *satisfying* (51) *and a* $p \times p$ *unitary complex matrix* u_1 *such that*

$$U = \operatorname{diag}(u_1, I_q)\begin{pmatrix} \Psi_1 & -\Psi_1\widehat{g^\#}\varrho_2^{-1}\Phi_2 \\ g & \Phi_2 \end{pmatrix}\operatorname{diag}(I_p, c_2). \tag{52}$$

Theorem 5.6. *Let the assumptions of Theorem 5.4 be satisfied. Then there exists a (p,q)-type Arov-inner function U which satisfies $U_{21} = g$ if and only if*

$$\widehat{g^{\#}\varrho_2^{-1}} \in [\mathcal{N}_+(\mathbb{D})]^{p\times q} \tag{53}$$

holds true. In this case, a given function $U \in [\mathcal{N}_+(\mathbb{D})]^{(p+q)\times(p+q)}$ is a (p,q)-type Arov-inner function which satisfies $U_{21} = g$ if and only if there are a $p \times p$ unitary complex matrix u_1 and a $q \times q$ unitary complex matrix u_2 such that

$$U = \operatorname{diag}(u_1, I_q)\begin{pmatrix} \Psi_1 & -\Psi_1\widehat{g^{\#}\varrho_2^{-1}}\Phi_2 \\ g & \Phi_2 \end{pmatrix}\operatorname{diag}(I_p, u_2). \tag{54}$$

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